SOUND AND MUSIC FROM CHUA'S CIRCUIT

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Nonlinear Dynamics have been very inspiring for musicians, but have rarely been considered specifically for sound synthesis. We discuss here the signals produced by Chua's circuit from an acoustical and musical point of view. We have designed a real-time simulation of Chua's circuit on a digital workstation allowing for easy experimentation with the properties and behaviors of the circuit and of the sounds. A surprisingly rich and novel family of musical sounds has been obtained. The audification of the local properties of the parameter space allows for easy determination of very complex structures which could not be computed analytically and would not be simple to determine by other methods. Finally, we have found that the time-delayed Chua's circuit can model the basic behavior of an interesting class of musical instruments.

1. Introduction
Nonlinear Dynamics have been very inspiring for musicians\(^1\) and have found interesting applications, for instance in the works of Pressing\(^2\) and Wessel.\(^3\) Pressing notes that several features make nonlinear maps potentially interesting as generators of musical design: fixed points, limit cycles, bifurcations, chaos and strange attractors. He has worked with the logistic map, a modified Metz map, the predator prey map and quadratic maps. The map output is used to control pitch selection, "interonset time", envelope attack time, dynamics, tempo, textural density and section length.

Wessel is using the output of Chua's circuit\(^4\) in the same way to set pitch, duration and dynamics. The remarkable trajectories from this circuit shown in Refs. 5 and 6 are very interesting candidates for melodies and rhythms. Some transformations are needed between the musician's intention and the parameters that operate on the nonlinear system. Wessel proposes using neural networks to learn and implement such transformations.

It seems that nonlinear dynamics have been less often considered specifically for sound synthesis.\(^7\) Here we examine sounds obtained by use of Chua's circuit.\(^8,9\) The
basic oscillator circuit (Fig. 1) contains three linear energy-storage elements (an inductor and two capacitors), a linear resistor, and a single nonlinear resistor $N_r$, called Chua's diode. The $v$–i characteristic $f$ of $N_r$ is shown in Fig. 2. The slopes on the different regions are designated by $m_0$, $m_1$ and $m_2$ respectively. Simple as it is, this circuit exhibits a surprisingly large variety of bifurcations and chaos.

2. Sound from Chua's Circuit

The state equations for Chua's circuit can be rewritten in a dimensionless form as

$$\dot{x} = \alpha [y - x - f(x)]$$
$$\dot{y} = z - y + x$$
$$\dot{z} = -\beta y .$$

For many values of the parameters $\alpha$ and $\beta$, the circuit exhibits stable orbits and the corresponding state variable values are periodic functions of time. As shown in Ref. 5 the circuit has interesting trajectories and periodic orbits. Similarly, the
signals of sustained sounds from musical instruments are generally periodic to a very good approximation. Therefore it is tempting to consider a state variable value as an acoustic signal to be amplified and sent to loudspeakers.\textsuperscript{13} As explained below, for more flexibility we have done a real-time simulation of the circuit on a digital computer with audio capabilities and we have designed a graphical-user interface to control it interactively.

According to the parameter values in Ref. 11, signals with period-1, period-2, etc ... can be obtained and lead to \textit{harmonic} sounds. In the case of chaotic signals that

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3}
\caption{Short-time spectrum of signals from the digital simulation of Chua's circuit for $m_0 = -1/7$, $m_1 = 2/7$, $\beta = 14.2857$ and for different values of alpha.}
\end{figure}
can be obtained with some other values, the corresponding sound can be qualified as noisy. A convenient way to characterize a sound is by the short-time spectrum of the signal, which, at least for sustained sounds, gives a rather good image of the way the sound is perceived. As an example, Fig. 3 shows the short-time spectrum of some signals from Chua's circuit for \( m_0 = -1/7, m_1 = 2/7, \beta = 14.2857 \) and for different values of \( \alpha \). These cases can be compared with the different regions displayed and commented on in Ref. 12, in particular in the frontispiece where the same parameter values are discussed. The results of the simple Euler simulation used here differ a little from the theoretical results, but very similar behaviors are found.

For \( \alpha = 8.0, 8.2 \) and 8.24 harmonic signals, hence harmonic sounds, are obtained with fundamental frequencies approximately 100 Hz (this unit is arbitrary as explained below), 50 Hz and 25 Hz respectively, corresponding to period-1, period-2 and period-4 in the dynamical system. For \( \alpha = 8.33, 8.36 \) and 8.5 noisy signals are obtained together with some sinusoidal components which are harmonics of the same fundamental frequencies as above. This simultaneous presence of sinusoidal components and noise in the signal is very interesting since this occurs for the majority of classical instruments and since this is relatively difficult to model in a way which is useful for musical purposes.\(^{14}\)

However, for musically interesting use, a synthesis algorithm has to provide control parameters allowing for expressive timbre modifications, i.e. essentially spectrum content modifications, as required by the performer. In order to get such flexibility, we now consider a slightly modified circuit, known as the time-delayed Chua's circuit\(^ {15}\) (in this issue).

3. Sound from the Time-Delayed Chua's Circuit

Let us look at Chua's circuit displayed in Fig. 1. Sharkovsky et al. add a DC bias voltage source in series with Chua's diode and replace the capacitor \( C2 \) and the inductance \( L \) by a lossless transmission line. The resulting time-delayed Chua's circuit is shown in Fig. 4. In a first simplification, the slopes \( m_0 \) and \( m_2 \) of the characteristic of \( N_x \) are set equal. The study of this dynamical system is difficult, but with \( C_1 = 0 \), it reduces to a nonlinear difference equation. The solution consists of the sum of an incident wave \( a(t - x/\nu) \) and a reflected wave \( b(t + x/\nu) \) such that

\[
a(t - x/\nu) = -b(t - x/\nu) = \Phi(t - x/\nu),
\]

and

\[
\Phi(t) = \gamma(\Phi(t - 2T)),
\]

where \( T \) is the time delay in the transmission line and \( \gamma \) is a piecewise-linear 1-D map which can be computed from the parameters of the circuit.\(^ {15}\) By a proper affine change of variable, the invariant interval of the map can be set to the interval \([0,1]\). For certain parameter values, the map \( \gamma \) is composed of two segments only in
the invariant interval, as shown on Fig. 5, with slopes $l_0$ and $l_1$. In this particular case, Sharkovsky et al. have shown analytically that the time-delayed Chua’s circuit exhibits the remarkable period-adding phenomenon shown in Fig. 11 of Ref. 15 (in this issue). In the $(l_0, l_1)$ space, the regions $\pi_2$, $\pi_3$, $\pi_4$, etc ... are those where the system has a stable limit cycle with periods respectively 2, 3, 4, etc ... In between every two consecutive stable regions the system has a chaotic behavior. The curves $(A, n)$ and $(S, n)$ are the limits of the stability regions.
Fig. 6. A simple time-delayed nonlinear system, also a basic clarinet model.

Fig. 7. Short-time spectrum of signals from the digital simulation of the time-delayed Chua's circuit for $l_0 = 0.99$ and $l_1$ between $-1$ and $-10000$. 
To simulate this very interesting circuit, we have implemented the digital system shown in Fig. 6. The role of the convolution by $h$ will be explained below, and for the moment we can ignore it. The time delay $T$ allows us to easily control the fundamental frequency of the produced sounds. As explained below, this delay is directly related to the structure and physics of many classical musical instruments. A very large variety of sounds can be produced by the system, and this is due to the combination of the rich dynamics of the nonlinear map together with the number of states represented by the delay line $T$. As an example, one can hear remarkable sounds by the use of $l_0 = 0.99$ and $l_1$ between $-1$ and $-10000$. Figure 7 shows the short-time spectrum of signals from the digital system for some of these values. Other values are discussed below.

4. Audification of the Structure of the Parameter Space

The structure of the periodic and chaotic regions in the $(l_0, l_1)$ space as displayed in Fig. 11 of Ref. 15 (in this issue) is interesting from a sonic point of view. The analytical computation is possible because the characteristic of the nonlinear element is piecewise-linear. The computation would not be possible for more complex characteristics. But by listening to the sound of the circuit, one can easily determine these regions and their frontiers. Let us take as an example the values for which histograms have been represented in Ref. 15, i.e. $l_1 = -18$ and $l_0$ varies from 0.04 to 0.49 or more. One can listen to the sound while changing parameter $l_0$. In $\pi_n$ regions, the periodic signal is clearly heard as a harmonic sound and the changes in periodicity are easily found by ear. In the intermediate chaotic regions, the sound is unstable or even noisy and it is not difficult to find approximate values for the frontiers between these regions.

It is remarkable that this audification of the local properties of the space allows an easy determination of very complex structures which in some cases cannot be computed analytically and are not simple to determine by other ways.

5. Comparison with Physical Models of Instruments

We now show that the time-delayed Chua's circuit is a model of an interesting class of musical instruments. Many physical models of sustained musical instruments have been proposed (strings, brass, reeds, flutes and voice). Mathematically, these models are dynamic systems which can be described by autonomous retarded functional difference and differential equations such as

$$x'(t) = f(x(t), x(t - T))$$  \hspace{1cm} (1)$$

or

$$x(t) = f(h \otimes x(t - T))$$  \hspace{1cm} (2)$$

where $f$ is a nonlinear function, $T$ some time delay, $h$ a convolution kernel and $\otimes$ the convolution operator.
For strings, reed–woodwinds or brass, the delay term plays an essential role. In clarinet models the instrument itself is represented by a delay line and the nonlinear excitation is represented by a time-varying pressure- or velocity-controlled reflection coefficient. Similarly, in violin models also (e.g. at the Institut de Recherche et de Coordination Acoustique/Musique, IRCAM, Paris) the string is set into oscillation by the bow and the combination is a Nonlinear Oscillator (NLO). A model for the lips of the trumpet player has also been implemented. Lips are represented by an NLO with two degrees of freedom, moved by pressure from the mouth and the mouthpiece.

The time delay found in the previous instrumental models comes from the instrument itself which is relatively easy to measure or estimate and can be modeled rather accurately as a linear system, by using, for instance, a state-space representation. One of the other key points for music synthesis is modeling the excitation process. This combination of nonlinear oscillators coupled to passive linear systems is a general model of a large class of musical instruments.

Let us examine the reed of a clarinet-like instrument coupled to the bore. Following Ref. 21, let us call $q_o$ and $q_i$ the outgoing and incoming pressure waves in the bore respectively, $p$ the pressure in the player’s mouth and $z$ the characteristic impedance of the bore. The system can be described in a simplified way by the equations

$$q_o(t) - q_i(t) = zF(q_i(t) + q_o(t) - p(t))$$

$$q_i = r(t) \otimes q_o(t) = h(t) \otimes q_o(t - T)$$

where $h(t - T) = r(t)$ is the reflection function of the bore. The most important assumption here is that the reed has no mass, leading to a memoryless nonlinearity $F$. In the case where this system has a unique solution, then

$$q_o = \gamma(h \otimes q_o(t - T)).$$

This is a very simple model to explain the basic oscillatory behavior of the reed in a clarinet-like instrument (Fig. 6). To better understand this behavior, McIntyre and Magenza note that if $h(t)$ is simplified into a Dirac impulse generalized function $\delta_t$ (the sign inversion is included in $\gamma$) then

$$q_o(t) = \gamma(q_o(t - T))$$

and similarly for $q_i$. The signal value $q_o(t)$ depends only on the value at $t - T$. If $q_o(t) = Q_o$ is constant on $[-T, 0]$ then it is constant on any interval $[(n - 1)T, nT]$ with a value

$$Q_n = \gamma(Q_{n-1}).$$
We now examine this iterated map as the basic model of a clarinet-like instrument. If \( p = 0 \), then \( q_0 \) should stay zero. Thus the origin \( O \) is a fixed point of the map. In order for the system to oscillate around \( O \), as we expect a musical instrument to do, the slope \( s_1 \) of a smooth map about \( O \) has to be less than \(-1\) (Fig. 8). In order for the signal not to grow to infinity, the slope of \( \gamma \) has to become greater than \(-1\) at some distance from \( O \). Let us choose \( \gamma \) as two segments with slopes \( s_1 < -1 \) and \( s_2 > -1 \) for \( Q > 0 \) and \( \gamma \) symmetric around \( O \) (Fig. 8). Remarkably, this is the same map as in the time-delayed Chua’s circuit. The \( s_1 - s_2 \) map is also justified by control considerations of the basic clarinet-like instrument as explained below. We thus have shown that the time-delayed Chua’s circuit is a model of an interesting class of musical instruments, namely those, like the clarinet, consisting of a massless reed coupled to a linear system.

![Fig. 8. A piecewise linear map with slopes \( s_1 \) and \( s_2 \).](image)

6. Roles of the Slopes \( s_1 \) and \( s_2 \)

It can easily be seen that \( |s_1| \) controls the transient onset velocity, the greater \( |s_1| \), the faster the onset. We have here a clear control parameter for the onset behavior of our instrument. If \( h(t) = \delta \), the signal is a square wave. If \( h(t) \) is a low pass kernel, then the signal is rounded. This rounding can be controlled by \( |s_2| \): the closer \( |s_2| \) is to unity, the less high frequencies are in \( q_0 \). This can also be viewed as follows: in the square wave case the system uses only two points of the map and in the rounded case it uses more points spread more regularly on the map.

As a first result, we have found that two important characteristics of the sound, transient onset velocity and richness, are controlled by the slopes \( s_1 \) and \( s_2 \). If one wants a smooth map, one may choose a simple cubic \( \gamma(x) = ax^3 + s_1 x \) (Fig. 9), where \( a \) is determined again according to the slopes \( s_1 \) in \( O \) and \( s_2 \) at the point of abscissa \( x_0 \) such that: \(-x_0 = ax_0^3 + s_1 x_0\).
7. Periodicity and Harmonic Content

For musical purposes we expect to have control of the period of the wave form since its inverse is the pitch of the sound. In the continuous case, Chow et al. have studied similar equations of the form

$$x(t) = f(h(t) \otimes x(t - T)).$$

It is shown that under some fairly general conditions on the map $f$ and the kernel $h$, period 2 (corresponding to the first mode in a clarinet, i.e. to a period of $2T$) is asymptotically stable. This means that we can expect to play and keep some steady tone from the instrument. It is also shown that, if $f$ is symmetrical around $O$ the signal $x(t)$ has the symmetry $x(t + T) = -x(t)$. Then the signal is composed of odd harmonics only. This is an essential characteristic of the clarinet sound. Under some conditions, periods having durations which are integer fractions of $2T$ are also possible.

As explained above, if the slope $s_2$ is greater than 1, the fixed point of the map $f^2 = f \circ f$ becomes unstable. We observe period doubling and, for greater values, chaotic behavior. From the sound synthesis point of view, this is very interesting. Period doubling corresponds to subharmonics. In the case of chaos, the signal sounds like noise added to the periodic tone of the instrument but with some relationship between partials and noise.

8. Digital Simulation and User Interface

For more flexibility, we have used a simulation of the time-delayed Chua’s circuit on a digital computer. We have implemented our simulation on a Silicon Graphics
Indigo workstation which is very well adapted for that purpose. It has good quality 16-bit audio ports and good graphic capabilities for user interface (Xwindow and Motif). Furthermore, it is fast enough for real-time simulation of the various circuits that we have studied. In order to keep up with real-time at a reasonable sampling frequency (such as 22050 Hz) integration is done by means of the simple forward Euler method. For given parameter values, the circuit has a precise period duration. We may allow arbitrary time stretching or compression in order to obtain any period duration, i.e. any pitch, wanted for the acoustic signal.

Real-time simulations were implemented using HTM, a tool for rapid prototyping of musical sound synthesis algorithms and control strategies. We have written a Motif-C++ graphical-user interface allowing for easy experimentation with the parameter values. An example of such an interface is shown on Fig. 10. The faders control the two slopes of the piecewise-linear map $\gamma$, the output amplitude and the fundamental frequency. The extreme values of the faders can be adjusted by editing the corresponding character strings. Buttons allow one to start the dynamical system by opening the data channel to the synthesis program, to display a graph of the output signal or of the map and to monitor the signal amplitude (meter).

9. Conclusion
We have studied the signals produced by Chua's circuit from an acoustical and musical point of view. We have designed a real-time simulation of Chua's circuit on
an affordable digital workstation allowing for interactive changes of the parameters, for simultaneous listening of the corresponding sounds and for easy experimentation with the properties and behaviors of the circuit and of the sounds. Very interesting results have been found. First, in the different regions of the parameter space, periodic and chaotic signals provide a rich and novel family of musical sounds.

It is easy to trace and characterize these different regions by listening to the sound while changing parameters. This audification of the local properties of the parameter space allows an easy determination of very complex structures which could not be computed analytically and would not be simple to determine by other ways.

Finally we have found that the time-delayed Chua’s circuit is a model of the basic behavior of an interesting class of musical instruments, namely those, like the clarinet, consisting of a massless reed coupled to a linear system. Such models are essential for the development and musical use of physical models of classical or new instruments. We expect to extend Chua’s circuit to other instruments such as brass, voice, flute, and strings.

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References